

**ATENISI UNIVERSITY**  
**2005**  
**Algebra 200 (Linear Algebra 1)**  
**Assignment 1**

Due: 4th April

Question 1:

Consider the system of linear equations

$$\begin{aligned}2x_1 + 5x_2 + 8x_3 &= 2 \\ x_1 + 2x_2 + 3x_3 &= 4 \\ 3x_1 + 4x_2 + 4x_3 &= 1\end{aligned}$$

- (a). Write down the augmented matrix for this system
- (b). Reduce the augmented matrix by elementary row operations to row echelon form.
- (c). Use your answer in part (b). to solve the above-system of linear equations.

Question 2:

Consider the system of linear equations

$$\begin{aligned}x_1 + \lambda x_2 - x_3 &= 1 \\ 2x_1 + x_2 + 2x_3 &= 5\lambda + 1 \\ x_1 - x_2 + 3x_3 &= 4\lambda + 2 \\ x_1 - 2\lambda x_2 + 7x_3 &= 10\lambda - 1\end{aligned}$$

- (a). Reduce its associated augmented matrix to row echelon form.
- (b). Find the value of  $\lambda$  for which the system is soluble.
- (c). Solve the system.
- (d). Consider the following matrices:

$$A = \begin{bmatrix} 2 & 5 \\ 1 & 4 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 7 & 2 & 9 \\ 9 & 2 & 7 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 4 \\ 2 & 1 & 3 \\ 1 & 1 & 3 \\ 3 & 2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 7 \\ 2 & 1 & 2 \\ 1 & 3 & 0 \end{bmatrix}$$

Calculate all possible products.

Question 3.

- (i). Evaluate  $A^2$ , where  $A = \begin{bmatrix} 2 & -5 \\ 3 & 1 \end{bmatrix}$ , find  $\alpha, \beta, \gamma \in \mathbf{R}$ , not all zero, such that the matrix

$\alpha I + \beta A + \gamma A^2$  is the zero matrix.

- (ii). Find all  $2 \times 2$  matrices  $B = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$  such that  $B^2$  is the zero matrix.

- (iii). Prove that if  $A$  and  $B$  are matrices such that  $I - AB$  is invertible, then the inverse of  $I - BA$  is given by  $(I - BA)^{-1} = I + B(I - AB)^{-1}A$ . [Hint: Write  $C = (I - AB)^{-1}$ . Then show that  $(I - BA)(I + BCA) = I$ ]

Question 4:

(a). Let  $A = (a_{ij})$ ,  $i = 1, \dots, m$  and  $j = 1, \dots, n$  be an  $m \times n$  matrix. Let  $B = (b_{jk})$ ,  $j = 1, \dots, n$  and  $k = 1, \dots, s$  be an  $n \times s$  matrix. Let  $AB = C$ . Show that the  $k$ -th column

$$C^k = b_{1k}A^1 + \dots + b_{nk}A^n.$$

(b). Use row elementary operation to find the inverse of matrix  $A$  if

$$A = \begin{bmatrix} 1 & a & b+c \\ 1 & b & a+c \\ 1 & c & a+b \end{bmatrix}$$

(c). Let  $A$  be a diagonal matrix:

$$A = \begin{bmatrix} a_1 & 0 & \dots & \dots & 0 \\ 0 & a_2 & 0 & \dots & 0 \\ \cdot & 0 & 0 & a_3 & 0 & \dots & 0 \\ \cdot & 0 & 0 & \cdot & & & \\ \cdot & 0 & 0 & & \cdot & & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_n \end{bmatrix}$$

If  $a_i \neq 0$  for all  $i$ , show that  $A$  is invertible. What is its inverse.

(d). Let  $A$  be a triangular matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \cdot & & a_{33} & \dots & \cdot \\ \dots & \dots & \dots & \dots & \cdot \\ 0 & \dots & 0 & 0 & \cdot & a_{n-1,n} \\ 0 & \dots & & 0 & 0 & a_{nn} \end{bmatrix}$$

Assume that no diagonal element is zero, and let

$$B = \begin{bmatrix} a_{11}^{-1} & a_{12} & \dots & a_{1n} \\ 0 & a_{22}^{-1} & \dots & a_{2n} \\ \cdot & \cdot & \dots & \cdot \\ \dots & \dots & \dots & \cdot \\ 0 & & a_{n-1,n-1}^{-1} & a_{n-1,n} \\ 0 & \dots & \dots & a_{nn}^{-1} \end{bmatrix}$$

Show that  $BA$  and  $AB$  are triangular matrices with components 1 on the diagonal.

Question 5:

(a). Let  $A$  be  $n \times n$  matrix. Define the **trace** of  $A$  to be the sum of the diagonal elements.

Thus if  $A = (a_{ij})$  then

$$\text{tr}(A) = \sum_{i=1}^n a_{ii}$$

Prove that  $\text{tr}(AB) = \text{tr}(BA)$  and  $\text{tr}(A) = \text{tr}(A^t)$ , where  $A^t$  is the transpose of  $A$ .

(b). For  $2 \times 2$  matrices show that  $(AB)^t = B^t A^t$

(c). A matrix  $B$  is an orthogonal matrix if  $I = B^t B$ . Show that  $B$  is orthogonal if

$$B = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{bmatrix}$$

(d). Show that if  $A$  is an orthogonal matrix then

$$(A\bar{u}) \bullet (A\bar{v}) = \bar{u} \bullet \bar{v},$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \bar{u} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \quad \text{and} \quad \bar{v} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}.$$

$\bullet$  is the symbol for dot product defined as: if  $\bar{a} = (a_1, a_2) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ ,  $\bar{b} = (b_1, b_2) = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$  then

$$\bar{a} \bullet \bar{b} = a_1 b_1 + a_2 b_2$$

**Question 6:**                      ANSWER ONLY ONE OF THE FOLLOWING

(i). Consider  $A$  to be strictly upper triangle matrix. Show that  $A^n = 0$ . (If you wish, you may do it in case  $n = 2, 3$  and  $4$ . The general case can be proved by induction).

(ii). Let

$$A = \begin{bmatrix} 1 & a_{12} & \dots & a_{1n} \\ 0 & 1 & \dots & a_{2n} \\ \cdot & & 1 & \dots \\ \dots & \dots & \dots & 1 \\ 0 & & & 1 & a_{n-1,n} \\ 0 & & & & 1 \end{bmatrix}$$

a triangular matrix with components 1 on the diagonal. Let  $N = A - I_n$ . Show that  $N^{n+1} = 0$ . Note that  $A$  is invertible, and that its inverse is

$$(I + N)^{-1} = I - N + N^2 - \dots + (-1)^n N^n.$$

(iii). If  $N$  is a square matrix such that  $N^{r+1} = 0$  for some positive integer  $r$ , show that  $I - N$  is invertible and that its inverse is  $I + N + \dots + N^r$ .