ATENISI UNIVERSITY

DEPARTMENT OF MATHEMATICS
Pure and Applied Mathematics

Calculus 100L

Examination

SEMESTER 1, 2005

Time Allowed: 3 Hours

Total Marks: 100

INSTRUCTIONS:

- There are 4 questions for this paper
- Attempt *all* questions
- Marks for each question are shown
- Show all working as partly correct answers may be rewarded
- Non-programmable calculators may be allowed.
- Write legibly

Question 1:

(a). Show that if

$$0 \le a \le b \quad \text{then} \quad \frac{a}{1+a} \le \frac{b}{1+b}.$$

5 Marks

(b). (i). Find the domain and range of $f(x) = \sqrt{9-x^2}$. **3 Marks**

(ii). Find $f(x)$ if $f(3x) = \dfrac{x}{x^2+1}$ [*Hint*: Let $z = 3x$ and find $f(z)$.] **4 Marks**

(c). (i). State the definition of a function $f(x)$ continuous at point $x = c$. **3 Marks**

(ii). Use the $\varepsilon - \delta$ definition of limit to show that $\lim\limits_{x \to 2} 3x - 5 = 1$ **5 Marks**

(d). Use the Pinching Theorem to determine $\lim\limits_{x \to 0} x^2 \sin^2 \dfrac{1}{x}$. **5 Marks**

$[$**25 Marks** $= \mathbf{5} + \mathbf{3} + \mathbf{4} + \mathbf{3} + \mathbf{5} + \mathbf{5}]$

Question 2:

(a). Find

$$(i).\ \lim_{x \to 2} \frac{x^2 + x + 1}{x^2 + 2x} \qquad (ii).\ \lim_{x \to -2} \frac{(x^2 - x - 6)^2}{(x+2)^2}$$

6 Marks

(b). (i). Evaluate $\lim\limits_{x \to \infty} \dfrac{\sqrt{9x^2 + 4x + 5}}{x}$ **4 Marks**

(ii). Find $\lim\limits_{x \to +\infty} \dfrac{c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n}{d_0 + d_1 x + d_2 x^2 + \ldots + d_m x^m}$, where $c_n \ne 0,\ d_m \ne 0$ **5 Marks**

(c). Show that $\lim\limits_{x \to 0} \left(\dfrac{1}{x} - \dfrac{1}{x^2} \right) = -\infty$ **5 Marks**

(d). Let $y = \dfrac{x}{x-9}$. Find $\dfrac{dy}{dx}$ using the definition of derivative: $y' = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

5 Marks

$[$**25 Marks** $= 6 + 4 + 5 + \mathbf{5} + \mathbf{5}]$

<u>Question 3</u>:

(a). Given that $h(0)=3, h'(0)=2$ find $f'(0)$ if $f(x)=h(x)+\dfrac{x}{h(x)}$ **5 Marks**

(b). Evaluate $\dfrac{d}{dt}\left(t^3-\dfrac{t}{t^2-1}\right)$ **5 Marks**

(c). (i). Prove: if the function f is differentiable at x, then $\dfrac{d}{dx}\left[f^2(x)\right]=2f(x)f'(x)$

5 Marks

(ii). Prove: $(f.g)''(x)=f''(x)g(x)+2f'(x)g'(x)+f(x)g''(x)$

5 Marks

(d). Find $\left.\dfrac{dy}{dx}\right|_{(0,2)}$ if $\sin x+y^3=8$ **5 Marks**

$\left[\mathbf{25\ Marks}=5+5+5+\mathbf{5}+\mathbf{5}\right]$

<u>Question 4</u>:

(a). Consider the function $f(x)=x^{2/3}$ in the closed interval $[-1,1]$.

(i). Show that there is no number c such that $f'(c)=0$

(ii). Explain why the conclusion of Rolle's Theorem does not hold.

5 Marks

(b). Let $f(x)=x^5-10x^3+25x$.

(i). Find all the stationary points.

(ii). Determine the local minimum and maximum points

5 Marks

(c). Let $g(x)=\dfrac{x}{x^2-1}$. Determine whether $g(x)$ is always increasing or decreasing on $(-1,1)$.

5 Marks

(d). Let $f(x)=x^3+1$. Find all values of $c\in[1,2]$ such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

5 Marks

(e). Determine $\lim\limits_{x\to 0}\dfrac{\sin^2 x}{x(1+\cos x)}$ using L'Hopital Rule. **5 Marks**

$\left[\mathbf{25\ Marks}=5+5+5+\mathbf{5}+\mathbf{5}\right]$